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Crack-tip field for fast fracture of an elastic–plastic–viscoplastic material incorporated with quasi-brittle damage. Part 1. Large damage regime

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Abstract

An asymptotic analysis of the near-tip field is presented in terms of the coordinate perturbation technique for fast crack propagation in an elastic–plastic–viscoplastic material with damage. A damage variable is incorporated in the constitutive relation based upon the strain-equivalence principle of damage mechanics. The damage evolution law used is a quasi-brittle type, in which both equivalent and hydrostatic stresses are involved. A non-singular stress field is obtained, as the damage has such a substantial influence on the material behaviour that the high stresses are relaxed at the crack tip. An analytical expression is obtained which explicitly shows the variation of stresses approaching the crack tip, and numerical computations of the angular distributions of stresses and strains are also presented. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

There have been many practical cases in which cracks propagate increasingly rapidly and eventually lead to large-scale unstable growth in engineering structures. To preclude and/or control such catastrophic occurrence, many investigations have been performed to understand the dynamic cracking process and to develop methodologies of crack arrest. For example, Amazigo and Hutchinson (1977) first investigated quasi-static crack growth in an elastic–plastic solid, while Hui and Riedel (1981) investigated quasi-static

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crack growth in an elastic–viscoplastic medium. Studies on transient fracture, without considering inertia effects of materials, in elastic–plastic or elastic–viscoplastic media also include the works of Hawk and Bassani (1986), Hui (1986) and others (Bose and Castañeda, 1992, etc.). On the other hand, Achenbach and Kanninen (1978), Achenbach et al. (1981), Aboudi and Achenbach (1983), Freund and Douglus (1982), Lo (1983), Gao and Nemat-Nasser (1983), Gao et al. (1983), Leighton et al. (1987), Östlund and Gudmundson (1988), and Ponte Castañeda (1987), analysed fast fracture in elastic–plastic or elastic–viscoplastic media in which inertia effects were taken into account. However, in many of these studies the focus was placed on specific materials that could be described by bilinear, elastic–perfectly-plastic, power-law hardening stress–strain curves and J_2 rule. With these material laws, all the stress fields obtained are shown to have singular characteristics. Clearly, the singular stress fields predicted are anomalous to practice in which stresses with finite amplitude is expected, if other factors, such as yielding and blunting of the crack tip, are considered. In particular, highly concentrated stresses will inevitably introduce damage in the material, which, in turn, degrades the material and relaxes the stresses. Whether or not singular stresses can be developed at the crack tip depends on the extent of damage. There are several papers that address the effect of damage on stress fields of static cracks. For example, Liu et al. (1994), Li et al. (1988), Tvergaard (1986), and Bassani and Hawk (1990) used fully numerical or combined analytical–numerical procedures to analyse the influence of damage on the crack-tip field, while others (e.g., Gao, 1986; Lu et al., 1997; Lee et al., 1997) gave analytical estimates to the damage field around the crack tip. Since these analytical studies only addressed stationary or quasi-static crack problems, the purpose of this paper is to evaluate the fast fracture problem for a generalised damaged-elastic–plastic–viscoplastic material, highlighting the influence of damage on the stress field very close to the crack tip.

We employ a continuum damage model developed by Lemaitre (1992), and incorporate the damage effect in the constitutive equation in terms of the strain-equivalence theorem of damage mechanics (Lemaitre, 1985, 1992). As will be shown, with this material model, a non-singular stress field is obtained.

A coordinate perturbation technique is applied to perform an asymptotic analysis to the fast fracture problem under consideration. For a first order approximation this treatment gives exactly the same result as an eigenexpansion method (for example, Williams, 1956). However it is distinctly different when higher order terms are included (e.g., Lu and Lee, 1998). The coordinate perturbation technique could be more convenient since the higher order expansions give linear rather than non-linear differential or integral–differential equations. This method hence provides a complete representation of the asymptotic expansions, provided that higher order solutions exist.

The arrangement of this paper is as follows. In Section 2 we list the basic equations, including the equation of motion, the constitutive relation, and the damage evolution law. In particular, in this section we derive the damage law in a form that applies to the crack-tip condition. Section 3 is devoted to the asymptotic analysis of the near-tip field. Numerical computations for the angular variations of stresses and velocities are carried out in Section 4, and Section 5 gives concluding remarks.

2. Governing equation

2.1. Equations of motion

The stress and deformation fields of the material containing a moving crack are referred to a coordinate system whose origin, O, is located at the crack tip. The crack is in the (x_1, x_2)-plane where the x_3-axis coincides with the crack front and the x_1-axis is in the direction of crack advance. The relevant displacement components are $u_1(x_1, x_2, t)$ and $u_2(x_1, x_2, t)$, where t is time. In the moving coordinate system, a material derivative with respect to time gives

$$\overset{\bullet}{(\cdot)} = \left\{ \frac{\partial}{\partial t} - \dot{a}(t)\frac{\partial}{\partial x_1} \right\}(\,\cdot\,) \equiv \left\{ \partial_t - \dot{a}(t)\Delta_{x_1} \right\}(\,\cdot\,), \tag{1}$$

where $\dot{a}(t)$ is the crack-tip speed. For convenience, hereafter we denote

$$\partial_t(\,\cdot\,) \equiv (\,\cdot\,)_{,t}.$$

For plane stress problems, the non-vanishing stress components are σ_{11}, $\sigma_{12}(=\sigma_{21})$ and σ_{22}. Thus, the equation of motion is

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} \quad (i,j = 1,2), \tag{2}$$

where ρ is the mass density of the material.

Unless otherwise specified the summation convention for repeated subscripts is used throughout the paper.

2.2. Constitutive relations

For some ductile metals at elevated temperatures, when a suddenly applied load is sufficient to produce an instantaneous plastic state in a zone, the total response of the media within this zone may include simultaneously elastic, plastic and viscoplastic effects. In this case, the constitutive relation of an elastic–plastic–viscoplastic material without damage can be described by (Riedel, 1981; Kanninen and Popelar, 1985)

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^{\mathrm{e}}_{ij} + \dot{\varepsilon}^{\mathrm{p}}_{ij} + \dot{\varepsilon}^{\mathrm{vp}}_{ij}, \tag{3a}$$

in which

$$\dot{\varepsilon}^{\mathrm{e}}_{ij} = \frac{1+v}{E}\dot{s}_{ij} + \frac{1-2v}{3E}\dot{\sigma}_{kk}\delta_{ij}, \tag{3b}$$

$$\dot{\varepsilon}^{\mathrm{p}}_{ij} = \tfrac{3}{2}B_0\bar{\sigma}^{(1/N)-2}\dot{\bar{\sigma}}s_{ij}, \tag{3c}$$

$$\dot{\varepsilon}^{\mathrm{vp}}_{ij} = \tfrac{3}{2}B\bar{\sigma}^{n-1}\bar{\varepsilon}^{-q}_{\mathrm{vp}}s_{ij} \quad (i,j,k = 1,2), \tag{3d}$$

where $\dot{\varepsilon}_{ij}$ is total strain rate, $\dot{\varepsilon}^{\mathrm{e}}_{ij}$ elastic strain rate, $\dot{\varepsilon}^{\mathrm{p}}_{ij}$ plastic strain rate and $\dot{\varepsilon}^{\mathrm{vp}}_{ij}$ viscoplastic strain rate. s_{ij} is deviatoric stress and $\bar{\sigma}$ is equivalent stress defined by

$$\bar{\sigma} = \left[\tfrac{3}{2}s_{ij}s_{ij}\right]^{1/2} = \left[\sigma^2_{11} + \sigma^2_{22} - \sigma_{11}\sigma_{22} + 3\sigma^2_{12}\right]^{1/2}, \tag{4}$$

where E represents Young's modulus, v is Poisson's ratio, n viscoplasticity exponent, B a temperature-dependent material coefficient, N hardening exponent, and B_0 a material coefficient related to the yield stress that can be expressed by $\alpha\sigma_y^{1-1/N}/E$, α is a material constant and $\bar{\varepsilon}_{\mathrm{vp}}$ is the effective viscoplastic strain. q is a material constant for primary viscoplastic creep ($q > 0$), secondary viscoplastic creep ($q = 0$), or tertiary viscoplastic creep ($q < 0$).

Note that the rate of $\bar{\varepsilon}_{\mathrm{vp}}$, namely, $\dot{\bar{\varepsilon}}_{\mathrm{vp}}$, is defined as

$$\dot{\bar{\varepsilon}}_{\mathrm{vp}} \equiv \left[\tfrac{2}{3}\dot{\varepsilon}^{\mathrm{vp}}_{ij}\dot{\varepsilon}^{\mathrm{vp}}_{ij}\right]^{1/2}. \tag{5a}$$

Then, we have

$$\dot{\bar{\varepsilon}}_{\mathrm{vp}} = B\bar{\sigma}^n\bar{\varepsilon}^{-q}_{\mathrm{vp}} \tag{5b}$$

by using Eqs. (3d), (4) and the definition of $\dot{\varepsilon}_{\mathrm{vp}}$ given in Eq. (5a).

Integration of Eq. (5b) with respect to t gives

$$\bar{\varepsilon}_{\rm vp} = \left\{(1+q)B\int_{t_0}^{t}\bar{\sigma}^n\,{\rm d}\tau\right\}^{1/(1+q)}, \tag{6}$$

with $\bar{\varepsilon}_{\rm vp} = 0$ at $t = t_0$.

The constitutive relation, Eq. (3a), can thus be rewritten as

$$\dot{\varepsilon}_{ij} = \frac{1+v}{E}\dot{s}_{ij} + \frac{1-2v}{3E}\dot{\sigma}_{kk}\delta_{ij} + \frac{3}{2}B_0\bar{\sigma}^{(1/N)-2}s_{ij}\dot{\bar{\sigma}} + \frac{3}{2}B\bar{\sigma}^{n-1}s_{ij}\left\{(1+q)B\int_{t_0}^{t}\bar{\sigma}^n\,{\rm d}\tau\right\}^{-q/(1+q)}. \tag{7}$$

Using the strain-equivalence principle of damage mechanics (Lemaitre, 1985, 1992), the constitutive relations corresponding to the damaged material can be derived in the same way as for the virgin material described by Eq. (7) except that the usual stress is replaced by the effective stress. Hence,

$$\begin{aligned}\dot{\varepsilon}_{ij} = {} & \frac{1+v}{E}\left(\overset{\bullet}{s_{ij}}\right)_{\rm eff} + \frac{1-2v}{3E}\left(\overset{\bullet}{\sigma_{kk}}\right)_{\rm eff}\delta_{ij} + \frac{3}{2}B_0(\bar{\sigma})_{\rm eff}^{(1/N)-2}\left(s_{ij}\right)_{\rm eff}\left(\overset{\bullet}{\bar{\sigma}}\right)_{\rm eff} + \frac{3}{2}B(\bar{\sigma})_{\rm eff}^{n-1}\left(s_{ij}\right)_{\rm eff} \\ & \times\left\{(1+q)B\int_{t_0}^{t}(\bar{\sigma})_{\rm eff}^{n}\,{\rm d}\tau\right\}^{-q/(1+q)},\end{aligned} \tag{8}$$

where

$$(\,\cdot\,)_{\rm eff} = \frac{(\,\cdot\,)}{1-D} \quad \text{and} \quad \overset{\bullet}{(\cdot)}_{\rm eff} = \frac{\rm d}{{\rm d}t}(\,\cdot\,)_{\rm eff},$$

in which D is an isotropic damage variable ranging from 0 for the undamaged state to 1 for complete failure of the material element.

2.3. Evolution law of damage

Observations have revealed that a fast moving crack in solids could result in two types of damage, namely, brittle and quasi-brittle damage. Brittle damage usually occurs in those materials such as high strength quenched steels or ceramics and concrete, for which there is no measurable plastic strain up to failure at the meso-scale. For ductile materials described by Eq. (7), plasticity effects can be relatively large or, more likely, be apparently small at some macro- or even meso-scale when they are experiencing fast fracture. However, even for small plasticity, damage develops at the micro-scale as a localised phenomenon and plastic strains occur in these small damaged volumes in the vicinity of the crack tip. In this case quasi-damage mechanism applies. There are different forms of damage evolution law (Chaboche, 1988a,b; Krajcinovic, 1996), and here we employ the modified quasi-brittle damage model suggested by Lemaitre (1992). That is,

$$\frac{{\rm d}D}{{\rm d}t} = \frac{1}{2ES_D}\left[\frac{2}{3}(1+v)\bar{\sigma}^2 + \frac{1}{3}(1-2v)\sigma_{kk}^2\right]\frac{{\rm d}P}{{\rm d}t}, \tag{9}$$

where S_D is a material constant which characterises the damage strength, and $\dot{P} = {\rm d}P/{\rm d}t$ is equivalent plastic strain rate defined by

$$\dot{P} \equiv \left[\tfrac{2}{3}\dot{\varepsilon}_{ij}^{\rm p}\dot{\varepsilon}_{ij}^{\rm p}\right]^{1/2}. \tag{10}$$

From Eqs. (3c) and (10) we obtain

$$\dot{P} = B_0\left(\frac{\bar{\sigma}}{1-D}\right)^{(1/N)-1}\frac{\rm d}{{\rm d}t}\left(\frac{\bar{\sigma}}{1-D}\right). \tag{11}$$

Substituting Eq. (11) into Eq. (9) we have

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{B_0}{2ES_D}\left[\frac{2}{3}(1+v)\bar{\sigma}^2 + \frac{1}{3}(1-2v)\sigma_{kk}^2\right]\left(\frac{\bar{\sigma}}{1-D}\right)^{(1/N)-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\bar{\sigma}}{1-D}\right). \tag{12}$$

Let

$$\lambda = 1 - \frac{1}{N}. \tag{13}$$

Eq. (12) can be recast as

$$-\frac{1}{\omega^2}\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{B_0}{6ES_D}\left[2(1+v)(\bar{\sigma})_{\mathrm{eff}}^2 + (1-2v)(\sigma_{kk})_{\mathrm{eff}}^2\right](\bar{\sigma})_{\mathrm{eff}}^{-\lambda}\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\sigma})_{\mathrm{eff}}, \tag{14}$$

where $\omega \equiv 1 - D$. Integration of Eq. (14) yields

$$\omega^{(2-\lambda)}(1-\omega) = \frac{B_0}{6ES_D}\left[\frac{2(1+v)}{3-\lambda}(\bar{\sigma})^2 + \frac{(1-2v)}{1-\lambda}(\sigma_{kk})^2\right](\bar{\sigma})^{1-\lambda}, \tag{15}$$

with $\omega = 1$ (or $D = 0$) for $(\sigma)_{\mathrm{eff}} = 0$.

Since the material at the moving crack tip must correspond to the fully failed state, then in the immediate vicinity of the crack tip,

$$\lim_{r\to 0} D \to 1 \equiv \lim_{r\to 0} \omega \to 0. \tag{16}$$

That is,

$$D \approx 1 \text{ or equivalently, } \omega \approx 0. \tag{17}$$

Eq. (15) is reduced to

$$\omega = \left(\frac{B_0}{6ES_D}\right)^{1/(2-\lambda)}\left[\frac{2(1+v)}{3-\lambda}(\bar{\sigma})^2 + \frac{(1-2v)}{1-\lambda}(\sigma_{kk})^2\right]^{1/(2-\lambda)}(\bar{\sigma})^{(1-\lambda)/(2-\lambda)} \tag{18}$$

under condition (17).

3. Field equations for near-tip analysis

Let (r, θ) ($-\pi < \theta \leqslant \pi$, with symmetry about $\theta = 0$) be the plane coordinates centred at the moving crack tip. In the following, a coordinate perturbation technique is applied to obtain an asymptotic solution:

$$\dot{u}_k(r,\theta,t) = r^s \sum_{m=0}^{\infty} \dot{U}_k^{(m)}(\theta,t) r^m \tag{19}$$

and

$$\sigma_{ij}(r,\theta,t) = r^s \sum_{m=0}^{\infty} \Sigma_{ij}^{(m)}(\theta,t) r^m. \tag{20}$$

With Eq. (20) the deviatoric stress s_{ij} and σ_{kk} can be expressed by

$$\left\{s_{ij}(r,\theta,t), \sigma_{kk}(r,\theta,t)\right\} = r^s \sum_{m=0}^{\infty}\left\{s_{ij}^{(m)}(\theta,t), \Sigma_{kk}^{(m)}(\theta,t)\right\} r^m, \tag{21}$$

where

$$S_{ij}^{(m)}(\theta,t) = \Sigma_{ij}^{(m)}(\theta,t) - \tfrac{1}{3}\Sigma_{kk}^{(m)}(\theta,t)\delta_{ij}. \tag{22}$$

Note that the parameter, s, as well as the unknown functions, $\dot{U}_k^{(m)}(\theta,t)$ and $\Sigma_{ij}^{(m)}(\theta,t)$, are to be determined. The relations

$$\frac{\partial}{\partial x_1} \equiv \Delta_{x_1} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \tag{23}$$

and

$$\frac{\partial}{\partial x_2} \equiv \Delta_{x_2} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \tag{24}$$

will be used below.

Substitution of Eqs. (19) and (20) into the equation of motion (2) gives

$$\rho\Big\{\partial_t - \dot{a}(t)\Delta_{x_1}\Big\}\left[r^s\sum_{m=0}^{\infty}\dot{U}_i^{(m)}(\theta,t)r^m\right] = \Delta_{x_j}\left[r^s\sum_{m=0}^{\infty}\Sigma_{ij}^{(m)}(\theta,t)r^m\right] \quad (i,j=1,2). \tag{25}$$

By using Eqs. (23) and (24), Eq. (25) can be rewritten as

$$-\Big[\rho\dot{a}(t)L_1^{(s)}\dot{U}_i^{(0)}(\theta,t) + L_j^{(s)}\Sigma_{ij}^{(0)}(\theta,t)\Big]r^{s-1} + \sum_{m=1}^{\infty}\Big[\rho\dot{U}_{i,t}^{(m-1)}(\theta,t) - \rho\dot{a}(t)L_1^{(m+s)}\dot{U}_i^{(m)}(\theta,t) - L_j^{(m+s)}\Sigma_{ij}^{(m)}(\theta,t)\Big]r^{m+s-1} = 0 \quad (i,j=1,2), \tag{26}$$

in which $L_i^{(\mu)}$ ($i = 1$, 2) is the differential operators defined by

$$L_1^{(\mu)} = \mu\cos\theta - \sin\theta\frac{\partial}{\partial\theta} \quad \text{and} \quad L_2^{(\mu)} = \mu\sin\theta + \cos\theta\frac{\partial}{\partial\theta} \tag{27}$$

for $i = 1$ and 2, respectively.

Note that

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial\dot{u}_i}{\partial x_j} + \frac{\partial\dot{u}_j}{\partial x_i}\right) \quad (i,j=1,2). \tag{28}$$

Replacing $\dot{\varepsilon}_{ij}$ in constitutive relation (7) with Eq. (28), and then substituting Eqs. (19) and (20) into both sides of the resulting relation, we have

$$\begin{aligned}
&-\frac{1}{2}\Big[L_i^{(s)}\dot{U}_j^{(0)}(\theta,t) + L_j^{(s)}\dot{U}_i^{(0)}(\theta,t)\Big]r^{s-1} - \left[\dot{a}(t)L_1^{[s(1-\eta)]}\left(\frac{1+v}{E}\Lambda_{ij}^{(0)}(\theta,t) + \frac{1-2v}{3E}\Pi^{(0)}(\theta,t)\delta_{ij}\right)\right]r^{s(1-\eta)-1} \\
&- \left[\frac{3}{2}B_0\Phi_{ij}^{(0)}(\theta,t)\right]r^{s(1-\eta)(\lambda_2+1)-1} + \left[\frac{3}{2}B\Omega_{ij}^{(0)}(\theta,t)\right]r^{s(1-\eta)n_q} + \frac{1}{2}\sum_{m=1}^{\infty}\Big[L_i^{(s+m)}\dot{U}_j^{(m)}(\theta,t) \\
&+ L_j^{(s+m)}\dot{U}_i^{(m)}(\theta,t)\Big]r^{m+s-1} + \sum_{m=1}^{\infty}\frac{1+v}{E}\Big[\Lambda_{ij,t}^{(m-1)}(\theta,t) - \dot{a}(t)L_1^{[m+s(1-\eta)]}\Lambda_{ij}^{(m)}(\theta,t)\Big]r^{m+s(1-\eta)-1} \\
&+ \sum_{m=1}^{\infty}\frac{1-2v}{3E}\Big[\Pi_{,t}^{(m-1)}(\theta,t) - \dot{a}(t)L_1^{[m+s(1-\eta)]}\Pi^{(m)}(\theta,t)\Big]\delta_{ij}r^{m+s(1-\eta)-1} + \sum_{m=1}^{\infty}\frac{3}{2}B_0\Big[\Xi_{ij,t}^{(m-1)}(\theta,t) \\
&- \dot{a}(t)\Phi_{ij}^{(m)}(\theta,t)\Big]r^{m+s(1-\eta)(\lambda_2+1)-1} + \sum_{m=1}^{\infty}\frac{3}{2}B\Omega_{ij}^{(m)}(\theta,t)r^{m+s(1-\eta)n_q} = 0, \quad (i,j=1,2),
\end{aligned} \tag{29}$$

where

$$\eta = \frac{2N+1}{N+1}, \tag{30}$$

$$\lambda_2 = \frac{1}{N} - 2, \tag{31}$$

$$n_q = \frac{n}{1+q}. \tag{32}$$

The expressions of $\Lambda_{ij}^{(m)}$, $\Pi^{(m)}$, $\Phi^{(m)}$, $\Omega_{ij}^{(m)}$ and $\Xi^{(m)}$ in Eq. (29) are given in Appendix A.

Eqs. (26) and (29) provide the asymptotic equations needed to evaluate the unknowns, $\dot{U}_i^{(m)}$ and $\Sigma_{ij}^{(m)}$, $(i,j = 1,2;\ m = 1,2,3,\ldots)$. For first order approximation we have

$$-\left[\rho\dot{a}(t)L_1^{(s)}\dot{U}_i^{(0)}(\theta,t) + L_j^{(s)}\Sigma_{ij}^{(0)}(\theta,t)\right]r^{s-1} = 0 \tag{33a}$$

and

$$-\frac{1}{2}\left[L_i^{(s)}\dot{U}_j^{(0)}(\theta,t) + L_j^{(s)}\dot{U}_i^{(0)}(\theta,t)\right]r^{s-1} - \left[\dot{a}(t)L_1^{(s(1-\eta))}\left(\frac{1+v}{E}\Lambda_{ij}^{(0)}(\theta,t) + \frac{1-2v}{3E}\Pi^{(0)}(\theta,t)\delta_{ij}\right)\right]r^{s(1-\eta)-1}$$
$$-\left[\frac{3}{2}B_0\Phi_{ij}^{(0)}(\theta,t)\right]r^{s(1-\eta)(\lambda_2+1)-1} + \left[\frac{3}{2}B\Omega_{ij}^{(0)}(\theta,t)\right]r^{s(1-\eta)n_q} = 0. \tag{33b}$$

Clearly, from Eqs. (33a) and (33b) it follows that elasticity, plasticity, and viscoplasticity may play different roles when the crack tip is approached, depending on the value of s. Since the problem described by Eqs. (33a) and (33b) is so complex, it seems quite unlikely for us to provide a complete interpretation of the crack-tip behaviour. (It turned out (see, Riedel, 1981) that even for a quasi-static crack in a creeping body without damage, mechanisms involved are so complicated that only approximate estimations can be conducted.) Nevertheless, following the method employed by Hui and Riedel (1981), the local analysis performed below may provide some interesting results which enable us to identify which term(s) may conditionally play a dominant role in the near-tip field.

Eq. (33a) simply indicates that the inertia term and the stress have the same order with respect to r, and the equation is valid for any value of s. Hence, we begin our analysis from Eq. (33b). First, let us examine the terms of the (effective) elasticity and the (effective) plasticity in Eq. (33b). The former is of the order of $r^{s(1-\eta)-1}$ whilst the latter is of the order of $r^{s(1-\eta)(\lambda_2+1)-1}$. For the majority of metals (see, for example, Riedel, 1981) $N = 0.2$–0.3. Under this condition we have

$$(1-\eta) = 1 - \frac{2N+1}{N+1} = -\frac{N}{1+N} < 0 \quad \text{and} \quad 1+\lambda_2 = \frac{1}{N} - 1 > 1. \tag{34}$$

Therefore, when $r \to 0$, which one is more important depends on the sign of s. If $s > 0$, then $|s(1-\eta) - 1| < |s(1-\eta)(1+\lambda_2) - 1|$. If $s < 0$, then $|s(1-\eta) - 1| > |s(1-\eta)(1+\lambda_2) - 1|$. Obviously, $s > 0$ corresponds to a non-singular stress field, and $s < 0$ to a singular field. From damage mechanics, the terminal state of damage is rupture, and the moving crack tip corresponds to a fully failed material meso-volume. Since a fully failed material element cannot sustain load, stresses at the crack tip should vanish. In contrast to this, the strain (and strain rate) at the crack tip goes to infinity because rupture occurs there. With these considerations, a positive s is expected in our case. Thus, the term related to the effective elasticity in Eq. (33b) can always be ignored in comparison with the effective plasticity.

With regard to comparisons between the third and the fourth terms, that is, between the effective plasticity and the effective viscoplasticity terms in Eq. (33b), there exist three possibilities as follows.

Case 1: Suppose that the effective plasticity dominates the crack-tip field, and the influence of effective viscoplasticity can be neglected. Equating the exponents of r of the relevant terms in Eq. (33b), we have

$$s - 1 = s(1 - \eta)(\lambda_2 + 1) - 1, \tag{35}$$

which yields

$$s \equiv 0.$$

Clearly, according to Eqs. (19) and (20), the result, $s \equiv 0$, would lead to multiple values of stresses and velocities at the point of the crack tip. They are not acceptable mathematically and physically.

It should be pointed that a different but relevant limiting case is that the constitutive equation, Eq. (7), can be reduced to the problem of a crack propagating in a power-law hardening material, provided that both damage and the viscoplastic term are neglected. Gao and Nemat-Nasser (1983) and Gao et al. (1983) suggested that an asymptotic solution with a logarithm type of singularity be available. However, the result they obtained there is restricted to elastically incompressible solids. Besides, an additional condition that $\Sigma'(\theta) \equiv 0$, namely, $\Sigma(\theta)$ is a constant, has to be applied. Without imposing any additional assumptions, such a form of asymptotic solutions (19) and (20) does not exist mathematically for Case 1. Note that this conclusion obtained here is based on such analysis in which the deformation theory is available and elastic unloading effects are not considered. A further discussion relevant to this issue is carried out elsewhere (Lu et al., 2001a), in which the elastic unloading and (possible) plastic re-unloading processes are taken into account.

Case 2: Suppose that the effective plasticity plays an equivalent role as the effective viscoplasticity. Again, repeating the same procedure as performed in Case 1, we obtain

$$s - 1 = s(1 - \eta)(\lambda_2 + 1) - 1 = s(1 - \eta)n_q. \tag{36}$$

It is readily seen that the above equalities cannot be simultaneously satisfied with any value of s. We may thus conclude that the assumption in this case is not valid and must be ruled out.

Case 3: Suppose the effective viscoplasticity plays a dominant role in the crack-tip field while other terms can be neglected. Following the same procedure as in Case 1 and Case 2, we have

$$s - 1 = s(1 - \eta)n_q,$$

i.e.

$$s = \left[1 - n_q(1 - \eta)\right]^{-1} = \frac{1 + N}{1 + (n_q + 1)N}. \tag{37}$$

It is readily verified that Eq. (37) is consistent with all governing equations.

For a number of metals, typically $n = 4$–6 and $N = 0.2$–0.3. Hence, the value of s, determined by Eq. (37), is always positive, which, as addressed above, predicts a non-singular stress field as we expect. This is a result by virtue of the concept of damage mechanics, and it cannot be obtained with conventional fracture mechanics alone. We may understand the non-singular stress field from this point of view: According to damage mechanics, the initiation and propagation of a crack can be modelled through the evolution of a damaged zone surrounding the crack. When damage increases, the material degrades and stresses are relaxed. When damage reaches its critical value at a point, the corresponding material meso-volume element is completely failed and fracture occurs there. The crack continues to advance with the adjacent meso-volume element reaching its critical state of damage, and so forth. In this way, no stress singularity ever exists; nor does the stress intensity factor. Most previous studies on this topic (for example, Chow and Wang, 1988) used fully numerical approaches, such as the finite element method, to complete the modelling. However, even finite element methods do not work so easily due to the localisation of damage at its critical stage. The problem is no longer elliptic, and a localised bifurcation of the solution may occur. In particular, if classic finite elements are used, the geometry of a crack is necessarily presumed a priori, and convergence regarding the mesh size is not assured in some cases. In contrast to a fully numerical treatment, the analysis conducted above does not have such disadvantages. More importantly, it permits an analytical insight into the issue concerned, and makes some essential characteristics of the problem readily understood.

4. Formulations for numerical computations

With Eq. (37), Eqs. (33a) and (33b) provide five equations – two from Eq. (33a) and three from Eq. (33b) – for computing the θ-distributions of stresses and velocities $\dot{U}_1^{(0)}(\theta,t)$, $\dot{U}_2^{(0)}(\theta,t)$, $\Sigma_{11}^{(0)}(\theta,t)$, $\Sigma_{12}^{(0)}(\theta,t)$, and $\Sigma_{22}^{(0)}(\theta,t)$. For convenience of computation, a new equivalent equation is formulated in terms of Eq. (33b) which can be written as

$$\dot{U}_{1,\theta}^{(0)}(\theta,t) = \left[s\cos\theta\dot{U}_1^{\prime(0)}(\theta,t) - \tfrac{3}{2}\Omega_{11}^{(0)}(\theta,t)\right]\sin^{-1}\theta, \tag{38}$$

$$\dot{U}_{2,\theta}^{(0)}(\theta,t) = \left[-s\cos\theta\dot{U}_2^{\prime(0)}(\theta,t) + \tfrac{3}{2}\Omega_{12}^{(0)}(\theta,t)\right]\cos^{-1}\theta, \tag{39}$$

$$s\cos\theta\dot{U}_{2,\theta}^{(0)}(\theta,t) + s\sin\theta\dot{U}_{1,\theta}^{(0)}(\theta,t) - \sin\theta\dot{U}_2^{(0)}(\theta,t) + \cos\theta\dot{U}_1^{(0)}(\theta,t) = 3B\Omega_{12}^{(0)}(\theta,t), \tag{40}$$

where

$$(\cdot)' = (\cdot)_{,\theta} = \frac{\partial}{\partial\theta}(\cdot).$$

Substituting Eqs. (38) and (39) into Eq. (40) and then differentiating the new equation with respect to θ on both sides, we obtain

$$\begin{aligned} &s\cos\theta\dot{U}_{1,\theta}^{(0)}(\theta,t) + s\sin\theta\dot{U}_{2,\theta}^{(0)}(\theta,t) - \cos^2\theta\Omega_{11,\theta}^{(0)}(\theta,t) - \sin^2\theta\Omega_{22,\theta}^{(0)}(\theta,t) - \sin 2\theta\Omega_{12,\theta}^{(0)}(\theta,t) \\ &\quad = s\sin\theta\dot{U}_1^{(0)}(\theta,t) - s\cos\theta\dot{U}_2^{(0)}(\theta,t) - \sin 2\theta\Omega_{11}^{(0)}(\theta,t) + \sin 2\theta\Omega_{22}^{(0)}(\theta,t) + 2\cos 2\theta\Omega_{12}^{(0)}(\theta,t). \end{aligned} \tag{41}$$

Eq. (41) is not an independent but an equivalent equation which is used to replace Eq. (40) for ease of computation. Thus, with Eqs. (33a), (38), (39) and (41), five independent differential equations have been established for five unknowns.

For second order approximations, a set of asymptotic equations can be derived as,

$$\rho\dot{a}(t)L_1^{(s)}\dot{U}_1^{(1)}(\theta,t) + L_j^{(s)}\Sigma_{1j}^{(1)}(\theta,t) = \rho\dot{U}_{1,t}^{(0)}(\theta,t) \quad (j=1,2), \tag{42}$$

$$\rho\dot{a}(t)L_1^{(s)}\dot{U}_2^{(1)}(\theta,t) + L_j^{(s)}\Sigma_{2j}^{(1)}(\theta,t) = \rho\dot{U}_{2,t}^{(0)}(\theta,t) \quad (j=1,2), \tag{43}$$

$$\dot{U}_{1,\theta}^{(1)}(\theta,t) = \left[(1+s)\cos\theta\dot{U}_1^{(1)}(\theta,t) - \tfrac{3}{2}B\Omega_{11}^{(1)}(\theta,t)\right]\sin^{-1}\theta, \tag{44}$$

$$\dot{U}_{2,\theta}^{(1)}(\theta,t) = \left[-(1+s)\sin\theta\dot{U}_2^{(1)}(\theta,t) + \tfrac{3}{2}B\Omega_{22}^{(1)}(\theta,t)\right]\cos^{-1}\theta, \tag{45}$$

$$\begin{aligned} &(1+s)\cos\theta\dot{U}_{1,\theta}^{(1)}(\theta,t) + (1+s)\sin\theta\dot{U}_{2,\theta}^{(1)}(\theta,t) - \cos^2\theta\Omega_{11,\theta}^{(1)}(\theta,t) - \sin^2\theta\Omega_{22,\theta}^{(1)}(\theta,t) - \sin 2\theta\Omega_{12,\theta}^{(1)}(\theta,t) \\ &\quad = (1+s)\sin\theta\dot{U}_1^{(1)}(\theta,t) - (1+s)\cos\theta\dot{U}_2^{(1)}(\theta,t) - \sin 2\theta\Omega_{11}^{(0)}(\theta,t) + \sin 2\theta\Omega_{22}^{(0)}(\theta,t) + 2\cos 2\theta\Omega_{12}^{(0)}(\theta,t). \end{aligned} \tag{46}$$

Higher order asymptotic equations can be obtained with the same procedure.

Eqs. (33a), (38), (39) and (41) are a set of non-linear partial differential equations with respect to t and θ. It is impossible to obtain a general closed-form solution for them. Let

$$\dot{U}_k^{(0)}(\theta,t) = A^{(0)}(t)\,\dot{\tilde{U}}_k^{(0)}(\theta), \quad (k=1,2) \tag{47}$$

and

$$\Sigma_{ij}^{(0)}(\theta,t) = G^{(0)}(t)\,\tilde{\Sigma}_{ij}^{(0)}(\theta), \quad (i,j=1,2) \tag{48}$$

be the specific form of solution. Certainly, Eqs. (47) and (48) are restrictive and depend on particular loading cases (e.g., proportional loading) and specific crack movements. Nevertheless, with such solutions, we may obtain some essential understanding of the angular variations of stresses and velocities in the crack-tip field.

Substituting Eqs. (47) and (48) into Eqs. (33a), (38), (39) and (41), we have

$$\rho\dot{a}(t)A^{(0)}(t) = k_1^{(0)}G^{(0)}(t) \tag{49}$$

and

$$A^{(0)}(t) = k_2^{(0)}\frac{3B}{2}\left(\frac{B_0}{6ES_D}\right)^{-nN/(N+1)(1+q)}\left[G^{(0)}(t)\right]^{-nN/(N+1)}\phi_q(t), \tag{50a}$$

in which

$$\phi_q(t) = \left\{(1+q)B\int_{t_0}^{t}\left[G^{(0)}(\tau)\right]^{-nN/(N+1)}\mathrm{d}\tau\right\}^{-q/(1+q)}, \tag{50b}$$

where $k_1^{(0)}$ and $k_2^{(0)}$ are two constants that cannot be determined by the asymptotic analysis. From Eqs. (49) and (50a), the expressions $A^{(0)}(t)$ and $G^{(0)}(t)$ can be explicitly obtained as:

$$G^{(0)}(t) = \begin{cases} \left[\dfrac{3}{2}\dfrac{k_2^{(0)}}{k_1^{(0)}}B\rho\dot{a}(t)\right]^{(N+1)/(nN+N+1)}\left(\dfrac{6ES_D}{B_0}\right)^{nN/(nN+N+1)}, & q=0 \\ \left\{\left[G^{(0)}(t_0)\right]^{1-\eta_N} + \dfrac{C_{\mathrm{G}}(1-\eta_N)}{\lambda_N}(t-t_0)\right\}^{1/(1-\eta_N)}, & q\neq 0 \end{cases} \tag{51a}$$

and

$$A^{(0)}(t) = \frac{k_1^{(0)}}{\rho\dot{a}(t)}G^{(0)}(t), \tag{51b}$$

where

$$\lambda_N = -\frac{1+q}{q}\,\frac{nN+N+1}{N+1}, \tag{52a}$$

$$C_{\mathrm{G}} = \left[\frac{3}{2}B\rho\dot{a}(t)\left(\frac{k_2^{(0)}}{k_1^{(0)}}\right)\left(\frac{6ES_D}{B_0}\right)^{\frac{nN}{(N+1)(1+q)}}\right]^{-(1+q)/q}(1+q)B, \tag{52b}$$

$$\eta_N = \frac{1}{N+1}\left[\frac{(1+q)}{q}(nN+N+1)(N+1) - nN + N + 1\right]. \tag{52c}$$

Obviously, in the case of steady crack propagation ($\dot{a} = \text{constant}$) in secondary creeping solid ($q = 0$), both $A^{(0)}$ and $G^{(0)}$ do not depend upon time, and are only functions of the velocity of crack, $k_1^{(0)}$ and $k_2^{(0)}$, as well as the relevant material constants.

With Eqs. (51a) and (51b), the partial differential equations (33a), (38), (39) and (41) are correspondingly reduced to a set of ordinary differential equations with respect to θ. That is,

$$k_1^{(0)} L_1^{(s)} \dot{\tilde{U}}_1^{(0)}(\theta) + L_j^{(s)} \tilde{\Sigma}_{1j}^{(0)}(\theta) = 0 \quad (j = 1, 2), \tag{53}$$

$$k_1^{(0)} L_1^{(s)} \dot{\tilde{U}}_2^{(0)}(\theta) + L_j^{(s)} \tilde{\Sigma}_{2j}^{(0)}(\theta) = 0 \quad (j = 1, 2), \tag{54}$$

$$\dot{\tilde{U}}_1'^{(0)}(\theta) = \left(k_2^{(0)}\right)^{-1} \left[s \cos\theta \tilde{U}_1^{(0)}(\theta) - \tfrac{3}{2} \tilde{\Omega}_{11}^{(0)}(\theta) \right] \sin^{-1}\theta, \tag{55}$$

$$\dot{\tilde{U}}_2'^{(0)}(\theta) = \left(k_2^{(0)}\right)^{-1} \left[- s \sin\theta \tilde{U}_2^{(0)}(\theta) + \tfrac{3}{2} \tilde{\Omega}_{12}^{(0)}(\theta) \right] \cos^{-1}\theta, \tag{56}$$

$$\begin{aligned} & s\cos\theta \dot{\tilde{U}}_1'^{(0)}(\theta) + \sin\theta \dot{\tilde{U}}_2'^{(0)}(\theta) - \left(k_2^{(0)}\right)^{-1} \left[\cos^2\theta \tilde{\Omega}_{11}'^{(0)}(\theta) + \sin^2\theta \tilde{\Omega}_{22}'^{(0)}(\theta) + \sin 2\theta \tilde{\Omega}_{12}'^{(0)}(\theta) \right] \\ & \quad = s\sin\theta \dot{\tilde{U}}_1^{(0)}(\theta) - s\cos\theta \dot{\tilde{U}}_2^{(0)}(\theta) - \left(k_2^{(0)}\right)^{-1} \left[\sin 2\theta \tilde{\Omega}_{11}^{(0)}(\theta) - \sin 2\theta \tilde{\Omega}_{22}^{(0)}(\theta) - 2\cos 2\theta \tilde{\Omega}_{12}^{(0)}(\theta) \right]. \end{aligned} \tag{57}$$

Here, $\tilde{\Omega}_{ij}^{(0)}(\theta)$ $(i,j = 1,2)$ is identical to $\Omega_{ij}^{(0)}(\theta)$ but with $U_i^{(0)}(\theta, t)$ and $\Sigma_{ij}^{(0)}(\theta, t)$ replaced by $\tilde{U}_i^{(0)}(\theta)$ and $\tilde{\Sigma}_{ij}^{(0)}(\theta)$, respectively. In Eqs. (53)–(57), the partial derivative $\partial/\partial\theta$ becomes the ordinary derivative, namely,

$$(\,\cdot\,)' = \frac{\mathrm{d}}{\mathrm{d}\theta}(\,\cdot\,). \tag{58}$$

Eqs. (53)–(57) determine the first order θ-variation of velocities and stresses. Higher order asymptotic angular solutions can be obtained as follows. If we let

$$\dot{U}_k^{(1)}(\theta, t) = A^{(1)}(t) \dot{\tilde{U}}_k^{(1)}(\theta), \tag{59}$$

$$\Sigma_{ij}^{(1)}(\theta, t) = G^{(1)}(t) \tilde{\Sigma}_{ij}^{(1)}(\theta), \quad (i, j, k = 1, 2), \tag{60}$$

and substitute Eqs. (59) and (60) in Eqs. (42)–(46), we obtain

$$G^{(1)}(t) = \begin{cases} k_1^{(1)} \rho \dot{A}^{(0)}(t), & \text{when } \dot{A}^{(0)}(t) \text{ is not identical to zero} \\ G^{(1)}(t), & \text{when } \dot{A}^{(0)}(t) \text{ is identical to zero} \end{cases} \tag{61}$$

and

$$A^{(1)}(t) = \begin{cases} k_2^{(1)} \frac{G^{(1)}(t)}{\rho \dot{a}(t)}, & \text{when } \dot{A}^{(0)}(t) \text{ is not identical to zero} \\ k_1^{(1)} \frac{G^{(1)}(t)}{\rho \dot{a}(t)}, & \text{when } \dot{A}^{(0)}(t) \text{ is identical to zero,} \end{cases} \tag{62}$$

where $k_1^{(1)}$ and $k_2^{(1)}$ are two constants to be determined, and

$$\dot{A}^{(0)}(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t} A^{(0)}(t). \tag{63}$$

Consequently, the second order asymptotic approximation, Eqs. (42)–(46), becomes a set of ordinary differential equations with respect to θ. In this paper, however, only the first order solutions will be illustrated in Section 5. Note that we have tacitly assumed that $\dot{a}(t)$ has a continuous first derivative in the derivation of Eqs. (61)–(63).

From the results described by Eqs. (51a), (51b), (61) and (62), it can be seen that the first and second order of time-dependent amplitude factors, namely, $A^{(0)}(t)$, $A^{(1)}(t)$, $G^{(0)}(t)$ and $G^{(1)}(t)$, can be explicitly determined except the constants $k_1^{(0)}$, $k_2^{(0)}$, $k_1^{(1)}$ and $k_2^{(1)}$. In fact, if other higher order asymptotic solutions can be split into forms of expressions given by Eqs. (47) and (48), it can be readily verified that the higher order time-dependent amplitude factors, $A^{(m)}(t)$ and $G^{(m)}(t)$ ($m = 2, 3, \ldots$), can also be determined following the same procedure. The only problem is that it needs further studies to ascertain if such solutions satisfy the physical and mathematical requirements.

The corresponding boundary conditions are subject to the symmetric constraints at $\theta = 0$,

$$\dot{U}_2(0,t) = 0 \quad \text{and} \quad \Sigma_{12}(0,t) = 0, \tag{64}$$

and the traction free requirement at $\theta = \pi$,

$$\Sigma_{22}(\pi,t) = 0 \quad \text{and} \quad \Sigma_{12}(\pi,t) = 0. \tag{65}$$

Expanding Eqs. (64) and (65) in the asymptotic forms, we have

$$\dot{\tilde{U}}_2^{(0)}(0) = \tilde{\Sigma}_{12}^{(0)}(0) = 0, \tag{66}$$

$$\tilde{\Sigma}_{22}^{(0)}(\pi) = \tilde{\Sigma}_{12}^{(0)}(\pi) = 0. \tag{67}$$

Since for Eq. (37) there exists a "singularity" at $\theta = 0$, the physical regularity constraint at $\theta = 0$ requires

$$s\dot{\tilde{U}}_1^{(0)}(0) - \tilde{\Omega}_{11}^{(0)}(0) = 0, \tag{68}$$

Eq. (68) gives the fifth boundary condition.

Eqs. (33a), (37), (38), and (40) and the boundary conditions Eqs. (66)–(68) define a non-linear two-point boundary-value problem, which can be solved by either the shooting method or the relaxation approach. The relaxation method was applied in this study, since the shooting method may not work even if the initial estimations are quite close to the true solutions.

In integrating Eqs. (33a), (38), (39) and (41) it is convenient to rewrite them in the form

$$Y'(\theta) = G[Y(\theta), \theta], \tag{69}$$

where

$$Y' = \left[\dot{\tilde{U}}_1'^{(0)}, \dot{\tilde{U}}_2'^{(0)}, \tilde{\Sigma}_{11}'^{(0)}, \tilde{\Sigma}_{22}'^{(0)}, \tilde{\Sigma}_{12}'^{(0)}\right]^{\mathrm{T}} \tag{70}$$

and

$$G = [g_1, g_2, g_3, g_4, g_5]^{\mathrm{T}}. \tag{71}$$

The expressions of g_1, g_2, g_3, g_4, and g_5 are given in Appendix B. There, $k_1^{(0)} = 1$ and $k_2^{(0)} = 1$.

5. Discussion and some numerical results

It can be seen from Eq. (37) that the value of s, which governs the way of the stress approaching the crack tip, is independent of the boundary conditions. As shown, s is totally determined by n and N, and it is a monotonic function decreasing with increasing n and N values. In particular, s is always within the range 0.2–0.6 for the majority of engineering materials which have $n = 4$–6 and $N = 0.2$–0.3. Poisson's ratio, ν, may alter the angular variations of stress and velocity fields through Eqs. (38), (39) and (41). Other parameters will influence the amplitude of the stress and velocity distributions via the time-dependent amplitude factors, $A^{(0)}(t)$ and $G^{(0)}(t)$.

The constants $k_1^{(0)}$, $k_2^{(0)}$, $k_1^{(1)}$ and $k_2^{(1)}$, etc., cannot be determined by the asymptotic analysis and depend on the global solution. Note that these constants appear not only in the time-dependent amplitude factors but also in the differential equations for angular variations. Some previous studies (for example, Hui and Riedel, 1981) did not introduce the arbitrary constants or took them as unity in their analyses.

From Eqs. (49) and (50) it can be seen that $k_1^{(0)}$ and $k_2^{(0)}$ characterise the ratio of the inertia field and the stress field. For example, if $\dot{a} \to 0$, then $k_1^{(0)} \to 0$ and the inertia effect can be neglected.

The influence of the material constants, n, N, and v, on the angular variation of stress and velocity fields is implicit. Some typical computed results are illustrated with $k_1^{(0)}$ and $k_2^{(0)}$ taken as unity.

Fig. 1 gives the non-dimensional angular variation of stresses with $n_q = 3.5$, $N = 0.25$, and $v = 0.3$. Clearly, it can be seen that values of the stress $\tilde{\Sigma}_{11}^{(0)}(\theta)$, $\tilde{\Sigma}_{22}^{(0)}(\theta)$ and $\tilde{\bar{\Sigma}}^{(0)}(\theta)$ (angular equivalent stress) are maximal at $\theta = 0$ and then decrease with increasing θ, and reach their minima at $\theta = \pi$. The angular variation of the shear stress, $\tilde{\Sigma}_{12}^{(0)}(\theta)$, is not monotonic. At $\theta = 0$, $\tilde{\Sigma}_{12}^{(0)}(\theta)$ vanishes and gradually increases with increasing θ. But at a certain point, which depends on materials constants and loading cases, $\tilde{\Sigma}_{12}^{(0)}(\theta)$ begins to decrease with increasing θ and vanishes again at another point. Afterwards, $\tilde{\Sigma}_{12}^{(0)}(\theta)$ repeats in a similar way but with an opposite sign, and finally vanishes again at $\theta = \pi$.

Fig. 2 shows the angular distribution of strains corresponding to Fig. 1. Note that the angular variations of strains can be obtained through time integration of the strain rate and the angular variation is identified. We can see from Fig. 2 that, in contrast to variations of stresses, the strains, $\tilde{\varepsilon}_{11}^{(0)}(\theta)$, $\tilde{\varepsilon}_{22}^{(0)}(\theta)$, and $\tilde{\varepsilon}^{(0)}(\theta)$ are minimum at $\theta = 0$ and then increase with increasing θ, and finally reach their maxima at $\theta = \pi$.

Fig. 3 gives the angular variations of $\dot{\tilde{U}}_1^{(0)}(\theta)$ for $n_q = 3.5$, 6.0 while $N = 0.25$, $v = 0.3$ and $n_q = 3.5$ but $N = 0.4$ with $v = 0.3$, and Fig. 4 shows the corresponding angular variations of $\dot{\tilde{U}}_2^{(0)}(\theta)$. Both Figs. 3 and 4 show that the velocities, $\dot{\tilde{U}}_1^{(0)}(\theta)$ and $\dot{\tilde{U}}_2^{(0)}(\theta)$, are monotonic increasing function with respect to θ, but the former is in the negative direction and the latter is in the positive direction.

Fig. 5 illustrates the angular variations of $\omega^{(0)}(\theta)$ for $n_q = 3.5$, 6.0 while $N = 0.25$, $v = 0.3$ and $n_q = 3, 5$ but $N = 0.4$ with $v = 0.3$, respectively. Since $\omega = 1 - D$, we may consider ω as a measure of the undamaged material. Obviously, $\omega^{(0)}(\theta)$ is maximum at $\theta = 0$ and is minimum at $\theta = \pi$, hence the corresponding damage is minimum at $\theta = 0$ and maximum at $\theta = \pi$.

The present theory provides a theoretical basis for fully numerical computations to the problem of fast fracture in a damaged-elastic–plastic–viscoplastic solid. The theory is not immediately applicable to give quantitative details for general fast crack propagation because the analysis is focused on specific materials. However, the concept of a severe damage zone very near to the crack tip is of practical importance in fast

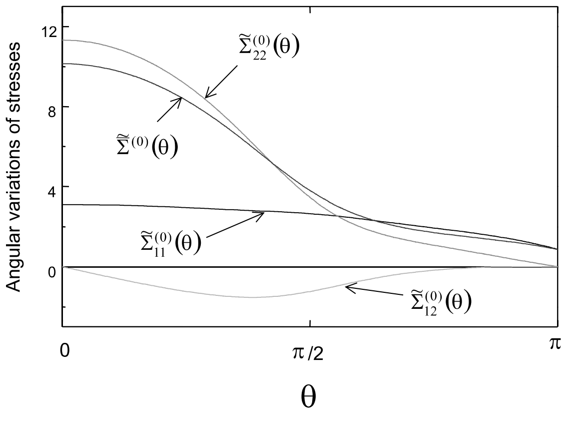

Fig. 1. Angular variation of stresses when $n_q = 3.5$, $N = 0.25$, and $v = 0.3$.

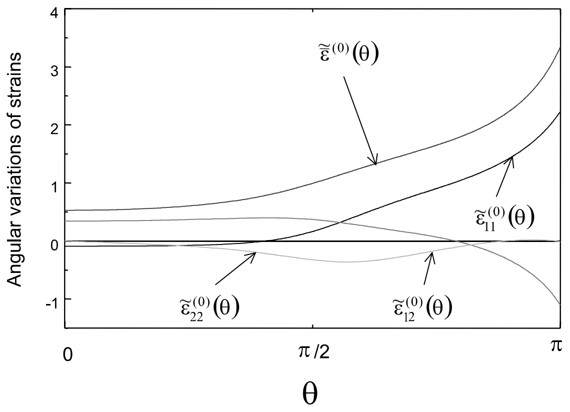

Fig. 2. Angular variation of strains when $n_q = 3.5$, $N = 0.25$, and $v = 0.3$.

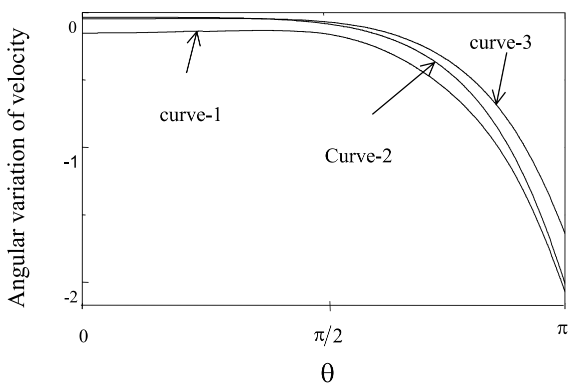

Fig. 3. Angular variation of $\tilde{U}_1^{(0)}(\theta)$; curve-1 – $n_q = 3.5$, $N = 0.25$, $v = 0.3$; curve-2 – $n_q = 6$, $N = 0.25$, $v = 0.3$; curve-3 – $n_q = 3.5$, $N = 0.4$, $v = 0.3$.

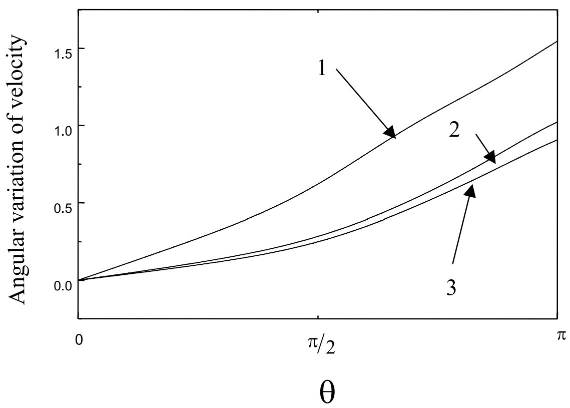

Fig. 4. Angular variation of $\tilde{U}_2^{(0)}(\theta)$; curve-1 – $n_q = 3.5$, $N = 0.25$, $v = 0.3$; curve-2 – $n_q = 6$, $N = 0.25$, $v = 0.3$; curve-3 –$n_q = 3.5$, $N = 0.4$, $v = 0.3$.

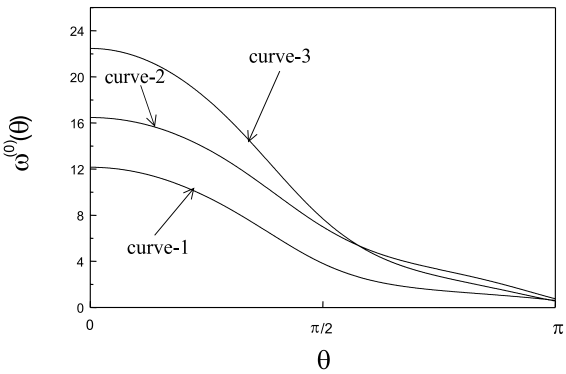

Fig. 5. Angular variation of $\omega^{(0)}(\theta)$; curve-1 – $n_q = 3.5$, $N = 0.25$, $\nu = 0.3$; curve-2 – $n_q = 6$, $N = 0.25$, $\nu = 0.3$; curve-3 – $n_q = 3.5$, $N = 0.4$, $\nu = 0.3$.

crack propagation testing. It should also be noted that $k_1^{(0)}$ and $k_2^{(0)}$ remain undetermined if we try to correlate branching and instability between the angular distributions of stress and velocity fields with a rupture criterion. How to determine $k_1^{(0)}$ and $k_2^{(0)}$ effectively and conveniently merits further studies.

6. Concluding remarks

An asymptotic solution has been obtained for the crack-tip stress field associated with fast fracture in an elastic–plastic–viscoplastic solid with quasi-brittle damage. The solution shows that stresses are not singular at the crack tip due to the damage effect. Hence, there exists a small zone around the crack tip where the prediction of singular stress field is no longer valid, since the damage effect dominates the tip behaviour there. In this case the damage is extremely severe and the damage variable D is close to unity. Another limiting case is that the damage can be relatively small and the damage variable, D, is much smaller than unity. For that case a parametric perturbation method is applied to analyse the tip behaviour that is described in Part 2 of this study (Lu et al., 2001b).

Note that in the zone where damage has a substantial influence on the crack-tip behaviour and the stress field is dominated by the effective viscoplasticity, the evolution of the quasi-damage itself is primarily controlled by plasticity effects as shown in Eq. (9), whilst the stress and strain rate fields are dominated by the effective viscoplasticity. In addition, when the inertia effect is included, the effective plasticity effect cannot be assumed to dominate the stress field of the fast moving crack tip alone. Otherwise, the asymptotic analysis would lead to either a trivial solution or mathematical inconsistence.

It should be pointed out that all the results obtained are based on the infinitesimal strain description. This starting point can be further improved since in the region very close to the crack tip, where damage is extremely severe, the small deformation description is not exactly accurate. Nevertheless, the present study provides a new asymptotic and analytical representation of the crack-tip behaviour, and establishes a means to understand the problem in a different way. Another point is that in the present analysis a total deformation theory has been used in which the unloading process is not involved. Thus, the zone concerned here is apparently limited to a sufficiently small scale where the full yielding condition is approximately satisfied and the total deformation description is acceptable. In another work (Lu et al., 2001a), we found that for fast fracture in the material described by Eqs. (3a)–(3d), where viscoplasticity is important, the elastic unloading domain is restricted to a very small sector ($<1°$) around the crack tip, provided that the crack velocity does not approach a limiting speed c_R, where c_R is the Rayleigh speed. Since the strain-equivalence principle of damage mechanics leads to the self-similarity of plasticity between the damaged medium and its virgin material, the result supports that the deformation description is applicable to our case. Note that with the development of damage in the material, the yielding threshold also decreases with softening of the material.

Acknowledgements

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Appendix A

From

$$\dot{u}_k(r,\theta,t) = r^s \sum_{m=0}^{\infty} \dot{U}_k^{(m)}(\theta,t) r^m \tag{A.1}$$

and

$$\sigma_{ij}(r,\theta,t) = r^s \sum_{m=0}^{\infty} \Sigma_{ij}^{(m)}(\theta,t) r^m. \tag{A.2}$$

Let

$$\hat{\dot{u}}_k(r,\theta,t) = \sum_{m=0}^{\infty} \dot{U}_k^{(m)}(\theta,t) r^m \tag{A.3}$$

and

$$\hat{\sigma}_{ij}(r,\theta,t) = \sum_{m=0}^{\infty} \Sigma_{ij}^{(m)}(\theta,t) r^m. \tag{A.4}$$

Then,

$$\Lambda_{ij}^{(m)}(\theta,t) = \frac{1}{m!} \frac{\partial^m}{\partial r^m} \left[\frac{\hat{s}_{ij}}{\hat{\omega}} \right] \Bigg|_{r=0}, \tag{A.5}$$

$$\Pi^{(m)}(\theta,t) = \frac{1}{m!} \frac{\partial^m}{\partial r^m} \left[\frac{\hat{\sigma}_{kk}}{\hat{\omega}} \right] \Bigg|_{r=0}, \tag{A.6}$$

$$\Gamma_{ij}^{(m)}(\theta,t) = \frac{1}{m!} \frac{\partial^m}{\partial r^m} \left[\frac{\hat{\bar{\sigma}}}{\hat{\omega}} \right]^{(1/N)-2} \left(\frac{\hat{s}_{ij}}{\hat{\omega}} \right) \Bigg|_{r=0}, \tag{A.7}$$

$$\Psi^{(m)}(\theta,t) = \frac{1}{m!} \frac{\partial^m}{\partial r^m} \left[\frac{\hat{\bar{\sigma}}}{\hat{\omega}} \right] \Bigg|_{r=0}, \tag{A.8}$$

$$\Omega_{ij}^{(m)}(\theta,t) = \frac{1}{m!}\frac{\partial^m}{\partial r^m}\left[\left(\frac{\hat{\bar{\sigma}}}{\hat{\omega}}\right)^{n-1}\frac{\hat{s}_{ij}}{\hat{\omega}}\right]\Bigg|_{r=0}, \tag{A.9}$$

$$\Xi_{ij}^{(m)}(\theta,t) = \sum_{q=0}^{m}\left[\Gamma_{ij}^{(q)}(\theta,t)\,\Psi_{,t}^{(m-q)}(\theta,t)\right], \tag{A.10}$$

$$\Phi_{ij}^{(m)}(\theta,t) = \sum_{q=0}^{m}\left[\Gamma_{ij}^{(q)}(\theta,t)\,L_1^{(m-q+s)}\,\Psi^{(m-q)}(\theta,t)\right]. \tag{A.11}$$

Here, the quantities with a hat '∧ ' have the same expressions as their original ones without the hat, except that the factor r^s has been removed.

Appendix B

Eqs. (32), (38), (39) and (41) can be rewritten in the matrix form

$$M(Y,\theta)\,Y' - F(Y,\theta) = 0, \tag{B.1}$$

where Y' is defined by Eq. (70) in the text, and

$$F = [f_1, f_2, f_3, f_4, f_5]^{\mathrm{T}} \tag{B.2}$$

in which

$$f_1 = s\cos\theta\,\dot{\tilde{U}}_1^{(0)}(\theta) + s\cos\theta\,\tilde{\Sigma}_{11}^{(0)}(\theta) + s\sin\theta\,\tilde{\Sigma}_{12}^{(0)}(\theta), \tag{B.3}$$

$$f_2 = s\cos\theta\,\dot{\tilde{U}}_2^{(0)}(\theta) + s\cos\theta\,\tilde{\Sigma}_{12}^{(0)}(\theta) + s\sin\theta\,\tilde{\Sigma}_{22}^{(0)}(\theta), \tag{B.4}$$

$$f_3 = s\cos\theta\,\dot{\tilde{U}}_1^{(0)}(\theta) - \tilde{\Omega}_{11}^{(0)}(\theta), \tag{B.5}$$

$$f_4 = \tilde{\Omega}_{22}^{(0)}(\theta) - s\sin\theta\,\dot{\tilde{U}}_2^{(0)}(\theta), \tag{B.6}$$

$$f_5 = s\sin\theta\,\dot{\tilde{U}}_1^{(0)}(\theta) - s\cos\theta\,\dot{\tilde{U}}_2^{(0)}(\theta) - \sin 2\theta\,\tilde{\Omega}_{11}^{(0)}(\theta) + \sin 2\theta\,\tilde{\Omega}_{22}^{(0)}(\theta) + 2\cos 2\theta\,\tilde{\Omega}_{12}^{(0)}(\theta), \tag{B.7}$$

while M is a matrix defined by

$$M = \begin{bmatrix} \sin\theta & 0 & 0 & \sin\theta & -\cos\theta \\ 0 & \sin\theta & 0 & -\cos\theta & \sin\theta \\ \sin\theta & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & 0 & 0 \\ s\cos\theta & s\sin\theta & n_1 & n_2 & n_3 \end{bmatrix}. \tag{B.8}$$

Here, n_1, n_2 and n_2 are represented by

$$n_1 = -\cos^2\theta\,C_{11}^{(1)} - \sin^2\theta\,C_{22}^{(1)} - \sin 2\theta\,C_{12}^{(1)}, \tag{B.9}$$

$$n_2 = -\cos^2\theta\,C_{11}^{(2)} - \sin^2\theta C_{22}^{(2)} - \sin 2\theta\,C_{12}^{(2)}, \tag{B.10}$$

$$n_1 = -\cos^2\theta\,C_{11}^{(3)} - \sin^2\theta\,C_{22}^{(3)} - \sin 2\theta C_{12}^{(3)}, \tag{B.11}$$

where

$$C_{11}^{(1)} = \frac{B_{11}^{(1)}}{2\tilde{\Sigma}^{(0)}} g_{12} + B_{11}^{(2)}, \quad C_{11}^{(2)} = \frac{B_{11}^{(1)}}{2\tilde{\Sigma}^{(0)}} g_{21} + B_{11}^{(3)}, \quad C_{11}^{(3)} = -\frac{3B_{11}^{(1)}}{\tilde{\Sigma}^{(0)}} \tilde{\Sigma}_{12}^{(0)}, \tag{B.12}$$

$$C_{22}^{(1)} = \frac{B_{22}^{(1)}}{2\tilde{\Sigma}^{(0)}} g_{12} + B_{22}^{(2)}, \quad C_{22}^{(2)} = \frac{B_{22}^{(1)}}{2\tilde{\Sigma}^{(0)}} g_{21} + B_{22}^{(3)}, \quad C_{22}^{(3)} = -\frac{3B_{22}^{(1)}}{\tilde{\Sigma}^{(0)}} \tilde{\Sigma}_{12}^{(0)}, \tag{B.13}$$

$$C_{12}^{(1)} = \frac{B_{12}^{(1)}}{2\tilde{\Sigma}^{(0)}} g_{12} + B_{12}^{(2)}, \quad C_{12}^{(2)} = \frac{B_{12}^{(1)}}{2\tilde{\Sigma}^{(0)}} g_{21} + B_{12}^{(3)}, \quad C_{12}^{(3)} = B_{12}^{(4)} - \frac{3B_{12}^{(1)}}{\tilde{\Sigma}^{(0)}} \tilde{\Sigma}_{12}^{(0)} \tag{B.14}$$

while

$$B_{11}^{(1)} = A_{11}^{(1)} - A_{11}^{(3)}\omega_3^{(0)}, \quad B_{11}^{(2)} = \tfrac{1}{3}\left(2A_{11}^{(2)} - A_{11}^{(3)}\omega_4^{(0)}\right), \quad B_{11}^{(3)} = -\tfrac{1}{3}\left(A_{11}^{(2)} + A_{11}^{(3)}\omega_4^{(0)}\right), \tag{B.15}$$

$$B_{22}^{(2)} = A_{22}^{(1)} - A_{22}^{(3)}\omega_3^{(0)}, \quad B_{22}^{(2)} = -\tfrac{1}{3}\left(A_{22}^{(2)} + A_{22}^{(3)}\omega_4^{(0)}\right), \quad B_{22}^{(3)} = \tfrac{1}{3}\left(A_{22}^{(2)} - A_{22}^{(3)}\omega_4^{(0)}\right), \tag{B.16}$$

$$B_{12}^{(1)} = A_{12}^{(1)} - A_{12}^{(3)}\omega, \quad B_{12}^{(2)} = -\tfrac{1}{3}A_{12}^{(3)}\omega_4^{(0)}, \quad B_{12}^{(3)} = -\tfrac{1}{3}A_{22}^{(3)}\omega_4^{(0)}, \quad B_{12}^{(4)} = \tfrac{1}{3}A_{12}^{(2)} \tag{B.17}$$

in which

$$A_{11}^{(1)} = H_1\tilde{S}_{11}^{(0)}, \quad A_{22}^{(1)} = H_1\tilde{S}_{22}^{(0)}, \quad A_{12}^{(1)} = H_1\tilde{S}_{12}^{(0)}, \tag{B.18}$$

$$A_{11}^{(2)} = H_2\tilde{S}_{11}^{(0)}, \quad A_{22}^{(2)} = H_2\tilde{S}_{22}^{(0)}, \quad A_{12}^{(2)} = H_2\tilde{S}_{12}^{(0)}, \tag{B.19}$$

$$A_{11}^{(3)} = H_3\tilde{S}_{11}^{(0)}, \quad A_{22}^{(3)} = H_3\tilde{S}_{22}^{(0)}, \quad A_{12}^{(3)} = H_3\tilde{S}_{12}^{(0)}. \tag{B.20}$$

In the above formulations,

$$g_{12} = 2\tilde{\Sigma}_{11}^{(0)} - \tilde{\Sigma}_{22}^{(0)},$$

$$g_{21} = 2\tilde{\Sigma}_{22}^{(0)} - \tilde{\Sigma}_{11}^{(0)},$$

$$\tilde{\Sigma}_{kk}^{(0)} = \tfrac{1}{3}\left[\tilde{\Sigma}_{11}^{(0)} + \tilde{\Sigma}_{22}^{(0)}\right],$$

$$\tilde{S}_{ij}^{(0)} = \tilde{\Sigma}_{ij}^{(0)} - \tfrac{1}{3}\tilde{\Sigma}_{kk}^{(0)}\delta_{ij},$$

$$\tilde{\Sigma}^{(0)} = \left[(\tilde{\Sigma}_{11}^{(0)})^2 + (\tilde{\Sigma}_{11}^{(0)})^2 - \tilde{\Sigma}_{11}^{(0)}\tilde{\Sigma}_{22}^{(0)} + 3(\tilde{\Sigma}_{12}^{(0)})^2\right]^{1/2},$$

$$\tilde{\omega}^{(0)} = \left[\frac{2(1+v)}{3-\lambda}(\tilde{\Sigma}^{(0)})^2 + \frac{(1-2v)}{1-\lambda}(\tilde{\Sigma}_{kk}^{(0)})^2\right]^{1/(2-\lambda)}[\tilde{\Sigma}^{(0)}]^{(1-\lambda)/(2-\lambda)},$$

$$\tilde{\omega}_1^{(0)} = \frac{1}{2-\lambda}\left[(1+v)(\tilde{\Sigma}^{(0)})^2 + (1-2v)(\tilde{\Sigma}_{kk}^{(0)})^2\right]^{(1+\lambda)/(2-\lambda)}[\tilde{\Sigma}^{(0)}]^{(1-\lambda)/(2-\lambda)},$$

$$\tilde{\omega}_2^{(0)} = \frac{1-\lambda}{2-\lambda}\left[(1+v)(\tilde{\Sigma}^{(0)}(\theta))^2 + (1-2v)(\tilde{\Sigma}_{kk}^{(0)}(\theta))^2\right]^{1/(2-\lambda)}[\tilde{\Sigma}^{(0)}(\theta)]^{(1+2\lambda)/(2-\lambda)},$$

$$\tilde{\omega}_3^{(0)} = 2(1+v)\tilde{\omega}^{(0)}\tilde{\omega}_1^{(0)}\tilde{\Sigma}^{(0)} + \tilde{\omega}_1^{(0)},$$

$$\tilde{\omega}_3^{(0)} = 2(1+v)\tilde{\omega}^{(0)}\tilde{\omega}_1^{(0)}\tilde{\Sigma}_{kk}^{(0)},$$

$$H_1 = (n_q - 1)(\tilde{\Sigma}^{(0)})^{n_q-1}(\tilde{\omega}^{(0)})^{-n_q},$$

$$H_2 = (\tilde{\Sigma}^{(0)})^{n_q-1}(\tilde{\omega}^{(0)})^{-n_q},$$

$$H_3 = -n_q(\tilde{\Sigma}^{(0)})^{n_q-1}(\tilde{\omega}^{(0)})^{-(n_q+1)}.$$

Recast Eq. (B.1) to

$$Y' = M^{-1}(Y,\theta)F(Y,\theta), \tag{B.21}$$

where M^{-1} is the inverse of M.

Let $G = M^{-1}F$ where $[g_1, g_2, g_3, g_4, g_5]^{\mathrm{T}}$. We obtain

$$g_1 = f_3 \sin^{-1}\theta, \tag{B.22}$$

$$g_2 = f_4 \cos^{-1}\theta, \tag{B.23}$$

$$\begin{aligned} g_3 = \big[& s\cos^3\theta \sin^{-1}\theta f_3 + (-\cos\theta f_5 - f_2 n_2 - f_1 n_3 + f_3 n_n)\cos\theta \\ & + (-f_1 n_2 + f_3 n_2 + f_4 n_2 + s\cos\theta f_4)\sin\theta\big](\Delta)^{-1}, \end{aligned} \tag{B.24}$$

$$\begin{aligned} g_4 = \big[& (s\sin\theta - n_3)\sin^2\theta\cos^{-1}\theta f_4 + (f_1 n_1 - f_3 n_1 - f_4 n_1 + f_2 n_3 - f_5 \sin\theta) \\ & + (f_2 n_1 + s\sin\theta f_3 \sin\,)\cos\theta](\Delta)^{-1}, \end{aligned} \tag{B.25}$$

$$\begin{aligned} g_5 = [& s\cos\theta(f_1 n_1 - f_3 n_1 + s\cos\theta f_2 n_2 + \sin\theta f_5)\cos\theta + (s\sin\theta f_4 - f_2 n_2)\sin\theta \\ & + f_4 n_2 \sin^2\theta\cos^{-1}\theta\big](\Delta)^{-1}, \end{aligned} \tag{B.26}$$

where

$$\Delta = -n_2\sin^2\theta - \cos\theta(\cos\theta n_1 + \sin\theta n_3). \tag{B.27}$$

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